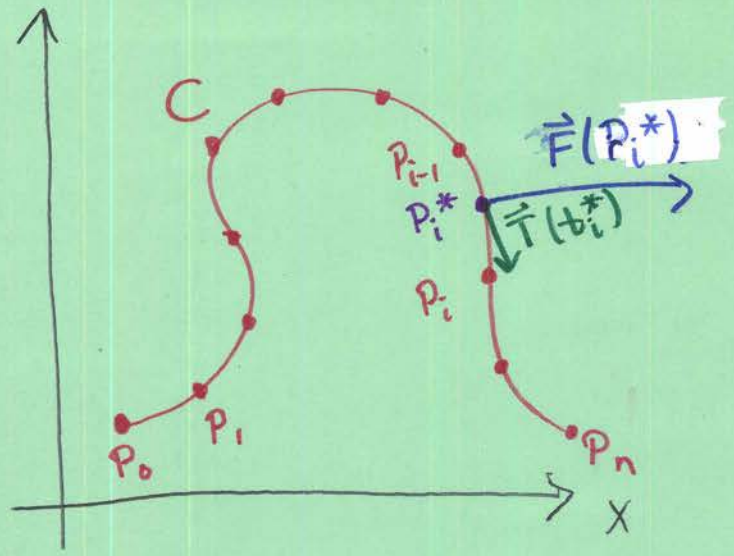


C parametrized by $\vec{r}(t)$, $a \le t \le b$.

The work performed over the segment from P_{i-1} to P_i is

$$W_i \approx \vec{F}(P_i^*) \cdot \vec{T}(t_i^*) \Delta s_i$$

giving, in the limit

$$\boxed{W = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(P_i^*) \cdot \vec{T}(t_i^*) \Delta s_i = \int_C \vec{F} \cdot \vec{T}\, ds}$$

Using known facts:

$$W = \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \left(\frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \right) (\cancel{\|\vec{r}\,'(t)\|}\, dt)$$

$$= \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_C \vec{F} \cdot d\vec{r}}$$

Lecture 17

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle xy, 3y^2 \rangle$ and C is parametrized by $\vec{r}(t) = \langle 11t^4, t^3 \rangle$, $0 \le t \le 1$.

Sol: $\vec{r}\,'(t) = \langle 44t^3, 3t^2 \rangle$, $\vec{F}(\vec{r}(t)) = \langle 11t^7, 3t^6 \rangle$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 484t^{10} + 9t^8.$$

So, $\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (484t^{10}+9t^8)\,dt = (44t^{11}+t^9)\Big|_0^1 = 45$ $\diamond$

Ex: Find the work done by $\vec{F} = \langle e^z, xz, x+y\rangle$ in moving a particle from the origin to $(1,1,-1)$ along $\vec{r}(t) = \langle t^2, t^3, -t\rangle$.

Sol: $\vec{r}(t) = \langle 0,0,0\rangle$ when $t=0$

$\vec{r}(t) = \langle 1,1,-1\rangle$ when $t=1$

$\vec{F}(\vec{r}(t)) = \langle e^{-t}, -t^3, t^2+t^3\rangle$, $\vec{r}\,'(t) = \langle 2t, 3t^2, -1\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 2te^{-t} - 3t^5 - t^2 - t^3$

$u=2t \quad dv=e^{-t}dt$

$W = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^1 (2te^{-t} - 3t^5 - t^2 - t^3)\,dt$

$= \left(-2te^{-t} - 2e^{-t} - \frac{1}{2}t^6 - \frac{1}{3}t^3 - \frac{1}{4}t^4\right)\Big|_0^1$

$= \left(-2e^{-1} - 2e^{-1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right) + 2 = -4e^{-1} - \frac{6}{12} - \frac{4}{12} - \frac{3}{12} + \frac{24}{12} = \frac{11}{12} - 4e^{-1}$ $\diamond$

16.3 - Fundamental Theorem of Line Integrals

Throughout the rest of the class, we will be generalizing the result so fundamental, it says so in its name: the fundamental theorem of calculus:

$$\int_a^b \frac{d}{dx} f(x)\,dx = f(b) - f(a).$$

There will be a theme to each of these: we will be trading derivatives for boundaries (or vice versa). Notice $\partial[a,b] = \{a,b\}$

Fundamental Theorem of Line Integrals

Let C be a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of 2 or 3 variables whose gradient is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

proof: $\nabla f = \langle f_x, f_y, f_z \rangle$

$$\int_C \nabla f \cdot dr = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \overset{\text{Chain rule}}{=} \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt$$

$$\overset{\text{FTOC}}{=} f(\vec{r}(b)) - f(\vec{r}(a))$$

$\square$

<u>Def</u>: We say a line integral $\int_C \vec{F}\cdot d\vec{r}$ is <u>independent of path</u> if given any two curves C_1, C_2 in the domain of $\vec{F}$ which start and end at the same place we have $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$.

The theorem tells us then that <u>line integrals of conservative vector fields are independent of path.</u>

<u>Def</u>: A curve is <u>closed</u> if it starts and ends at the same point.

<u>Thm</u>: $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for all closed paths C in D.

<u>Def</u>: A set D is <u>open</u> if for every point P in D, we can fit a disk of radius $\varepsilon > 0$ (ε as small as needed) around P inside D.

<u>Def</u>: A set D is <u>connected</u> if any two points in D can be joined by a path in D.

<u>Thm</u>: Suppose $\vec{F}$ is a vector field which is continuous on an open & connected set D. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is conservative on D.

Before going on for how to check whether a vector field is conservative, let's pick up some terminology and facts.

16.5 - Curl and Divergence

Def: The operator ∇ (prounced "del") is
$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

We've already seen one use:
$$\nabla f = \text{grad } f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

Def: Let $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ be a vector field.

- the divergence of $\vec{F}$ is
$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$
- the curl of $\vec{F}$ is
$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \vec{F}$$
$$= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

Notice that to find the curl, $\vec{F}$ must be a 3D vector field!

<u>Ex</u>: Find the curl and divergence of

$$\vec{F} = \langle xy^2z^3, x^3yz^2, x^2y^3z \rangle$$

<u>Sol</u>:

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$

$$= \langle 3x^2y^2z - 2x^3yz, 3xy^2z^2 - 2xy^3z, 3x^2yz^2 - 2xyz^3 \rangle$$

$$\text{div } \vec{F} = y^2z^3 + x^3z^2 + x^2y^3$$

◇

We have the useful facts:

<u>Thm</u>: Let $\vec{F} = \langle P, Q, R \rangle$ and $f = f(x,y,z)$ and suppose P, Q, R, and f are C^2. Then:

i) $\text{curl}(\text{grad } f) = \vec{0} = \text{curl}(\nabla f)$

ii) $\text{div}(\text{curl } \vec{F}) = \vec{0}$

The proof of each of these follow from Clairaut's theorem. Notice that part (i) says that the curl of a conservative vector field is $\vec{0}$.

Finally, we approach conservative vector fields:

16.3 & 16.5 - Conservative Vector Fields and Potentials

If $\vec{F} = \langle P, Q \rangle$ is conservative, then $P = f_x$ & $Q = f_y$ for some $f = f(x,y)$. So, by Clairaut's theorem $P_y = Q_x$ (as long as $\vec{F}$ is C^1). This gives us a test for conservative vector fields

Test: Let $\vec{F}$ be a C^1 vector field.

a) If $\vec{F}(x,y) = \langle P, Q \rangle$ and $\vec{F}$ is conservative, then

$$P_y = Q_x.$$

b) If $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ and $\vec{F}$ is conservative, then

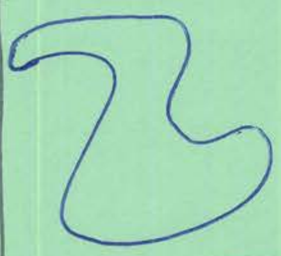
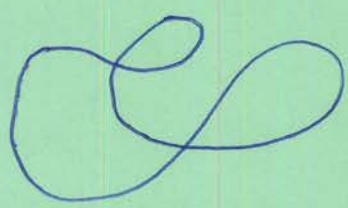
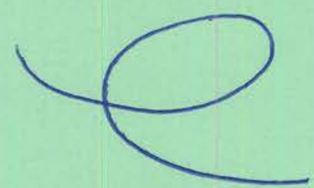
$$\text{curl}\,\vec{F} = \vec{0}.$$

These are tests we can use to determine whether $\vec{F}$ is not conservative. If $P_y \neq Q_x$ in (a) or $\text{curl}\,\vec{F} \neq \vec{0}$ in (b), then $\vec{F}$ is not conservative. If the equations hold, then we don't know if $\vec{F}$ is conservative or not.

A curiosity would be to know when these results are reversible, i.e., when $P_y = Q_x$ or $\text{curl}\,\vec{F} = \vec{0}$ implies $\vec{F}$ is conservative. To answer this, we need 2 definitions:

Def: A curve which does not cross itself anywhere between its endpoints is called simple.

Ex:

	simple	not simple
closed		
not closed		

Def: A region D is called simply connected if D is connected and every simple closed curve in D encloses only points in D.

Here are the reversals:

Theorem:

a) If $\vec{F} = \langle P, Q \rangle$ is a C^1 vector field on an open, simply connected region D, and $P_y = Q_x$, we have that $\vec{F}$ is conservative.

b) If $\vec{F} = \langle P, Q, R \rangle$ is a C^1 vector field on all of $\mathbb{R}^3$ and $\operatorname{curl} \vec{F} = \vec{0}$, then $\vec{F}$ is conservative.

Often, the easiest way to determine whether a vector field is conservative is to just try to find a potential.

Ex: Compute $\int_C (\ln y + 2xy^3)\,dx + (3x^2y^2 + \frac{x}{y})\,dy$

where C has parametric equations

$x = \frac{1}{2}t^2 + 2,\ y = e^t(1 + 2t - t^2),\ 0 \le t \le 2.$

Sol: Both the curve and the integrand look difficult, so, let's try to use the FToLI:

First, we need a potential for:

$$\vec{F} = \langle \ln y + 2xy^3,\ 3x^2y^2 + \frac{x}{y} \rangle = \langle P, Q \rangle$$

If $\vec{F}$ is conservative $\vec{F} = \langle f_x, f_y \rangle = \langle P, Q \rangle$, so

$$f = \int P\,dx = \int (\ln y + 2xy^3)\,dx = x\ln y + x^2y^3 + g(y)$$

$$f_y = \frac{x}{y} + 3x^2y^2 + g'(y) = Q = 3x^2y^2 + \frac{x}{y} \Rightarrow g'(y) = 0 \Rightarrow g(y) = K$$

$$\Rightarrow f = x\ln y + x^2y^3 + K$$

We can choose any value for C that we want since we just need __a__ potential. So, let's take $K = 0$. The endpoints of C are $(2,1)$ & $(4,e^2)$, so

$$\int_C \vec{F}\cdot d\vec{r} = f(4,e^2) - f(2,1) = (4\ln e^2 + 16e^6) - (2\ln 1 + 4)$$

$$= 8 + 16e^6 - 4 = 4 + 16e^6$$

$\diamond$

Let's do an application to end this section:

Let $\vec{F}$ be a continuous force field which moves an object along a path C given by $\vec{r}(t)$, $a \le t \le b$, where $\vec{r}(a) = A$ & $\vec{r}(b) = B$. By Newton's second law,

$$\vec{F}(\vec{r}(t)) = m\vec{r}\,''(t)$$

Thus, the work done by $\vec{F}$ is then

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_a^b m\vec{r}\,''(t) \cdot \vec{r}\,'(t)\,dt.$$

$$= \frac{m}{2}\int_a^b \frac{d}{dt}\left[\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right] dt = \frac{m}{2}\int_a^b \frac{d}{dt}\|\vec{r}\,'(t)\|^2\,dt$$

$$= \frac{1}{2}m\|\vec{r}\,'(t)\|^2 \Big|_a^b = \frac{1}{2}m\|\vec{r}\,'(b)\|^2 - \frac{1}{2}m\|\vec{r}\,'(a)\|^2$$

$$= \frac{1}{2}m\left(\|\vec{v}(b)\|^2 - \|\vec{v}(a)\|^2\right)$$

The quantity $\frac{1}{2}m\|\vec{v}(t)\|^2$ is the <u>kinetic energy</u> at time t, so we can write $W = K(B) - K(A)$. ①

Now, let's assume that the force $\vec{F}$ is conservative. Then $\vec{F} = \nabla f$. Typically, in Physics, the <u>potential energy</u> is defined as $P = -f$ so that $\vec{F} = -\nabla P$. Then, the FToLI gives

$$W = \int_C \vec{F} \cdot d\vec{r} = -\int_A^B \nabla P \cdot d\vec{r} = -\left(P(B) - P(A)\right) = P(A) - P(B) \quad ②$$

Combining ① & ②:

$$K(B) - K(A) = W = P(A) - P(B) \iff P(A) + K(A) = P(B) + K(B)$$

Law of Conservation of Energy